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Basic Analysis Techniques & Multi-Spacecraft Data — Computer Session —

Sheet 3 5 June 2007

3 Gradient estimation accuracy in model magnetic fields

The estimation of spatial derivatives (curl, divergence, gradient) is a key issue in Cluster data analysis. In the following you make use of the reciprocal vector method to estimate the spatial derivative matrix along simplified spacecraft orbits in given model magnetic fields. The inter-spacecraft distance can be varied as well as the magnetic field model and the imposed noise level. Since this exercise is meant to help assessing the imperfections of the gradient estimation method, the model magnetic fields are chosen to be curl-free (and, of course, divergence-free).

3.1 Model fields and spacecraft configuration

The model fields used in the IDL program `testkb` are

- the cylindrical magnetic field of a long straight wire (`curved`, circular field lines in planes perpendicular to the z-axis),
- a radial field in two dimensions (`rad2d`, straight but convergent field lines in planes perpendicular to z),
- a radial field in three dimensions (`rad3d`, radially convergent fieldlines), and
- a dipole field (`dipole`, both field line convergence and curvature are present).

All fields are divergence-free and curl-free everywhere outside the origin ($\mathbf{r} = 0$). The fields are given in cartesian coordinates in appendix 3.3.3.

Spatial derivatives of the model fields are evaluated using the reciprocal vector method, see 3.3.1. The method rests on the assumption of linear field variations. This implies that for the nonlinearly varying magnetic fields given above errors are introduced which should depend on the inter-spacecraft distance Δ relative to the inhomogeneity length scale L (curvature radius or convergence length scale). For the example fields studied here, L varies linearly with distance r, thus the ratio Δ/r should be indicative of the nonlinear error contribution.

Besides inaccuracies in spatial gradient estimation due to nonlinear field variations, there are also errors introduced by inaccurately determined spacecraft positions, and by measurement errors. The latter can be simulated in the IDL program `testkb` through additive noise.

3.2 How to use the IDL program `testkb`

All IDL programs needed for this exercise can be found in the subdirectory `ComputerSessions/BasicAnalysisTechniques_Vogt/ex3/` of the workshop web page[1]. For convenience, you may download the zip archive `ex3pro.zip` which contains all of them. You may want to create a subdirectory for this exercise on your local PC

```
Linux> mkdir ex3
Linux> cd ex3
```

move the zip archive to the subdirectory, and then unzip it

```
Linux> unzip ex3pro.zip
```

The driver routine is `testkb`. In order to use it, you simply edit the parameter section (either in an editor like `xemacs`, or using the `idlde` editor), and then type

```
IDL> .r testkb
```

at the IDL prompt.

3.2.1 Parameters

Here are the parameters that you should play with when you start using the IDL program `testkb`.

`BMODEL` The parameter selects one of the model fields implemented in the program to see the different effects of field line convergence or curvature on the resulting error.

`NOISEPARAM` Choose a small non-zero value to mimic the effects of additive noise on the measurements.

`BUNIT` If `BUNIT=1` is set, the magnetic field is normalized before gradient estimation so that effectively the elements of the matrix $\nabla\hat{\mathbf{B}}$ are estimated. This should yield measures of inhomogeneity length scales. Note that $|\hat{\mathbf{B}}\cdot\nabla\hat{\mathbf{B}}|^{-1}$ is the curvature radius, and $|\hat{\mathbf{V}}\cdot\nabla\hat{\mathbf{B}}|^{-1}$ can be understood as a convergence length scale if $\hat{\mathbf{V}}$ is perpendicular to the local magnetic field vector.

`SCALEDB` If this parameter is set to `SCALEDB=1`, then the output is scaled: radial distance with the inter-spacecraft distance Δ, and spatial derivatives are multiplied with the ratio Δ/B, where B is the strength of the local magnetic field.

The default values of the other parameters are reasonable and do not need to be changed if you just start to play around with this program.

The results are displayed as line plots in double-logarithmic representation. Different line styles indicated different inter-spacecraft distances. Graphics output can be directed to a postscipt file by means of the parameter `POSTSCRIPT`.

[1] http://www.faculty.iu-bremen.de/jvogt/cospar/cbw6/

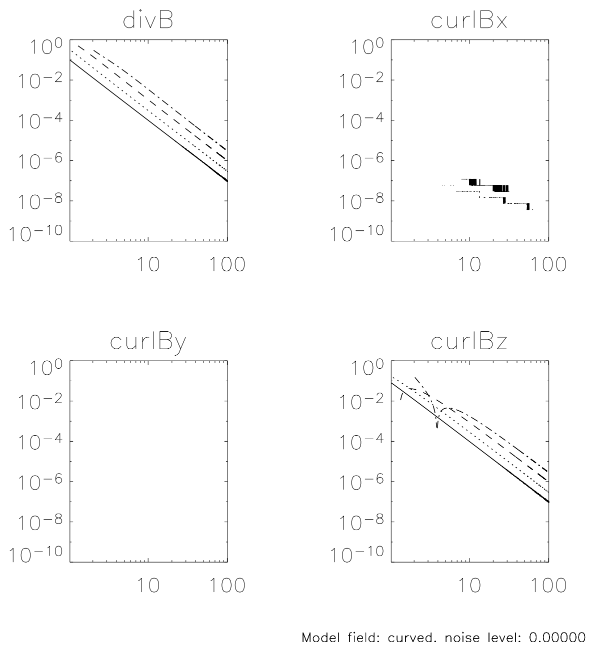

Figure 1: Estimation of divergence and curl in the (cylindrical) magnetic field of a long straight wire. The separation of lines reflects differences in the inter-spacecraft distances Δ. The estimated $\nabla \cdot \mathbf{B}$ and $(\nabla \times \mathbf{B})_z$ fall of with distance r as Δ/r^3.

3.2.2 Selected examples

You may start with `BMODEL='curved'`, and zero values for `NOISEPARAM`, `BUNIT`, and `SCALEDB`. You should get first the graphics in figure 1. The straight lines with a slope of -3 for the estimated values of $\nabla \cdot \mathbf{B}$ and $(\nabla \times \mathbf{B})_z$ indicate a power law dependence on r, and the separation of the four lines. suggest a (linear) dependence on inter-spacecraft distance Δ. (Note that the spacecraft distances are chosen to be 0.1, 0.3, 1.0, 3.0 in normalized units by default.) The figure is thus consistent with the analytical result presented in appendix 3.3.4, namely, that both $\nabla \cdot \mathbf{B}$ and $(\nabla \times \mathbf{B})_z$ should be proportional to Δ/r^3 for small Δ/r.

Changing the model to `BMODEL='rad3d'` gives a slightly different picture. The resulting graphics is shown in figure 2. The straight lines in the panels for $\nabla \cdot \mathbf{B}$, $(\nabla \times \mathbf{B})_y$, and $(\nabla \times \mathbf{B})_z$ fall off with distance r as Δ/r^4 for small Δ/r which is consistent with the results presented in appendix 3.3.4.

The effect of additive noise is shown in figure 3. The parameter `NOISEPARAM` was chosen to be 10^{-4}. The figure helps to understand when measurement inaccuracies become more important than nonlinear contributions to the field variations.

3.3 Appendix: Theory of spatial gradient estimation

So far (at least) two methods to estimate spatial derivatives from Cluster data have been introduced by various authors, namely, the so-called curlometer technique and the reciprocal vector / barycentric coordinates method). Both methods are equivalent in the sense that linear field

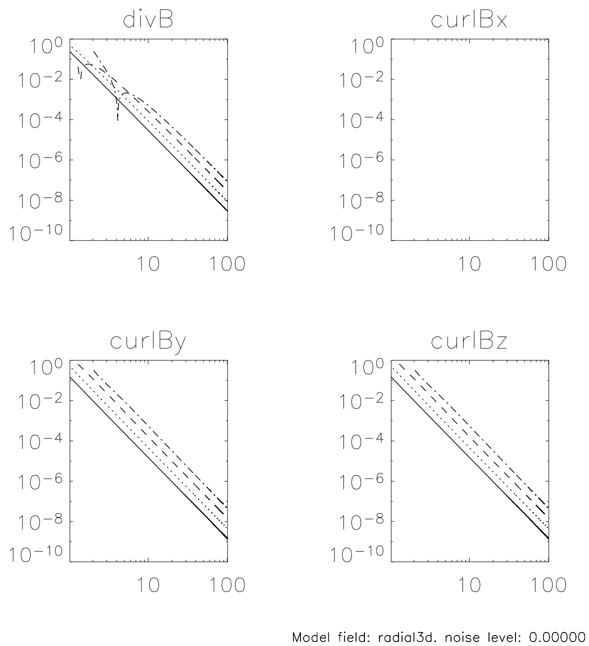

Figure 2: Estimation of divergence and curl in a radially converging field. The estimated $\nabla \cdot \mathbf{B}$, $(\nabla \times \mathbf{B})_y$, and $(\nabla \times \mathbf{B})_z$ fall of with distance r as Δ/r^4.

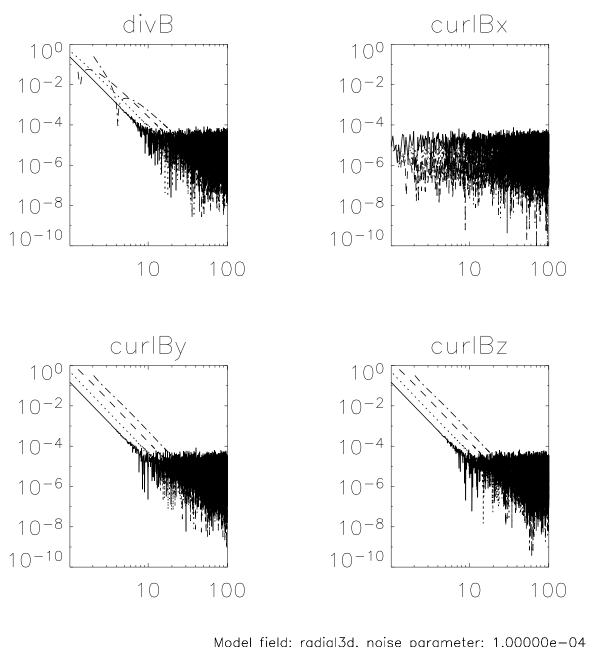

Figure 3: Estimation of divergence and curl in a radially converging field if noise is added. For small values of Δ/r the estimated $\nabla \cdot \mathbf{B}$, $(\nabla \times \mathbf{B})_y$, and $(\nabla \times \mathbf{B})_z$ are more affected by measurement errors than by nonlinear field variations.

variations are evaluated (note that the linear approximation of a field is unique if the values at four distinct points in space are known). Detailed discussions can be found in various chapters of *Analysis Methods for Multi-Spacecraft Data*, G. Paschmann and P. Daly (Eds.), ESA Publication Division (Noordwijk, Netherlands), 1998; hereafter referred to as the *ISSI Cluster data analysis book*. It is available as free pdf from the *International Space Science Institute (Bern, Switzerland)*. For the 6th COSPAR Capacity Building Workshop a local copy can be accessed from the workshop web page[2].

In this exercise we only address the reciprocal vector method because the structure of the estimation formulas appears to be more transparent.

3.3.1 The reciprocal vector method

Briefly, barycentric coordinates provide a convenient means to linearly interpolate a physical quantity g inside a satellite cluster tetrahedron by using the measured values g_α at the four spacecraft positions $\mathbf{r}_\alpha$:

$$\tilde{g}(\mathbf{r}) \;=\; \sum_{\alpha=0}^{3} \mu_\alpha(\mathbf{r}) g_\alpha$$

where:

$$\mu_\alpha(\mathbf{r}) \;=\; 1 \,+\, \mathbf{k}_\alpha \cdot (\mathbf{r} - \mathbf{r}_\alpha)$$

$\tilde{g}$ denotes the linear function that interpolates between the measurements. The vectors $\mathbf{k}_\alpha$ are given by the formula:

$$\mathbf{k}_\alpha \;=\; \frac{\mathbf{r}_{\beta\gamma} \times \mathbf{r}_{\beta\lambda}}{\mathbf{r}_{\beta\alpha} \cdot (\mathbf{r}_{\beta\gamma} \times \mathbf{r}_{\beta\lambda})}$$

$(\alpha, \beta, \gamma, \lambda)$ must be a permutation of $(0, 1, 2, 3)$. Relative position vectors are denoted by $\mathbf{r}_{\alpha\beta} = \mathbf{r}_\beta - \mathbf{r}_\alpha$. The set $\{\mathbf{k}_\alpha\}$ is called the reciprocal base of the tetrahedron.

Vector functions $\mathbf{B}$ can be handled in a similar way by applying the above formulas to the cartesian components. Since $\tilde{g}$ and $\tilde{\mathbf{B}}$ are linear functions, the calculation of the derivatives can be done quite easily. The results are:

$$\begin{aligned}
\nabla g \;\simeq\; \nabla \tilde{g} \;&=\; \sum_{\alpha=0}^{3} \mathbf{k}_\alpha g_\alpha \\
\hat{\mathbf{e}} \cdot \nabla g \;\simeq\; \hat{\mathbf{e}} \cdot \nabla \tilde{g} \;&=\; \sum_{\alpha=0}^{3} (\hat{\mathbf{e}} \cdot \mathbf{k}_\alpha) g_\alpha \\
\nabla \cdot \mathbf{B} \;\simeq\; \nabla \cdot \tilde{\mathbf{B}} \;&=\; \sum_{\alpha=0}^{3} \mathbf{k}_\alpha \cdot \mathbf{B}_\alpha \\
\nabla \times \mathbf{B} \;\simeq\; \nabla \times \tilde{\mathbf{B}} \;&=\; \sum_{\alpha=0}^{3} \mathbf{k}_\alpha \times \mathbf{B}_\alpha
\end{aligned}$$

The element (i, j) of the matrix $\nabla\mathbf{B}$ is given by:

$$\frac{\partial V_j}{\partial x_i} \;\equiv\; (\nabla\mathbf{B})_{ij} \;\simeq\; \sum_{\alpha=0}^{3} (\mathbf{k}_\alpha \mathbf{B}_\alpha)_{ij} \;\equiv\; \sum_{\alpha=0}^{3} k_{\alpha i} V_{\alpha j}$$

[2] http://www.faculty.iu-bremen.de/jvogt/cospar/cbw6/Etc/issibook.pdf

With regard to error estimation it is important to notice that $\nabla \times \mathbf{B}$ and $\nabla \cdot \mathbf{B}$ are just linear combinations of various $(\nabla\mathbf{B})_{ij}$'s and thus of terms like $k_{\alpha i}V_{\alpha j}$, with $i = j$ or $i \neq j$.

In short,

$$\nabla\mathbf{B} \;\simeq\; \nabla \cdot \tilde{\mathbf{B}} \;=\; \sum_{\alpha=0}^{3} \mathbf{k}_\alpha \mathbf{B}_\alpha^\dagger \ .$$

Note that the superscript † denotes the transposition (of a matrix, or, if applied to a vector, to turn column vectors into row vectors and vice versa).

3.3.2 Inter-spacecraft distance and geometric error parameters

In this exercise we assume (without loss of generality) that we analyse the data in a frame moving with the barycenter of the tetrahedron

$$\mathbf{r}^{\mathrm{bc}} \;=\; \frac{1}{4}\sum_{\alpha=0}^{3} \mathbf{r}_\alpha \ ,$$

thus $\mathbf{r}^{\mathrm{bc}} = 0$. Then the parameter

$$R \;=\; \sqrt{\frac{1}{4}\sum_{\alpha=0}^{3} |\mathbf{r}_\alpha|^2}$$

is a measure of the distance between the four Cluster spacecraft that can be used for scaling purposes. Note that R^2 is 1/4 the trace of the spacecraft position tensor

$$\mathsf{R} \;=\; \sum_{\alpha=0}^{3} \mathbf{r}_\alpha \mathbf{r}_\alpha^\dagger \ .$$

(This expression differs by a factor of 1/4 from the definition of the volumetric tensor given by Harvey in the ISSI Cluster data analysis book.)

In a similar manner we can define the reciprocal tensor as

$$\mathsf{K} \;=\; \sum_{\alpha=0}^{3} \mathbf{k}_\alpha \mathbf{k}_\alpha^\dagger$$

which can be shown to be the inverse of the spacecraft position tensor. The square root of the trace of K, i.e.,

$$K \;=\; \sqrt{\sum_{\alpha=0}^{3} |\mathbf{k}_\alpha|^2}$$

constitutes an inverse length scale which turns out to be important in the analysis of geometric errors in gradient estimation. First-order (isotropic) error estimation yields for the geometric error of a spatial derivative $D\mathbf{B}$:

$$\delta|D\mathbf{B}| \;=\; \sqrt{\frac{f}{3}}\, K\, \delta B$$

where δB denotes a typical error of the field measurement. The parameter f can be understood as the number of degrees of freedom of the differential operator D; use $f = 3$ if $D\mathbf{B} = \nabla \cdot \mathbf{B}$, $f = 2$ if $D\mathbf{B} = \nabla \times \mathbf{B}$, and $f = 1$ if $D\mathbf{B} = \hat{\mathbf{e}} \cdot \nabla\mathbf{B}$ (directional derivative or partial derivative). For details, see the ISSI Cluster data analysis book.

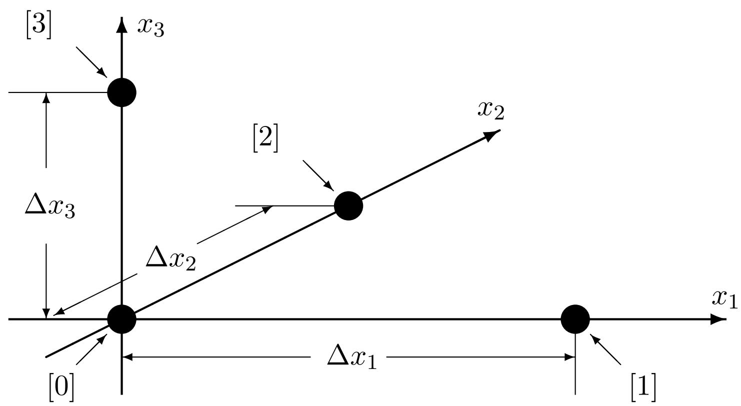

Figure 4: Sketch of the three-dimensional model satellite configuration used in the IDL program `testkb`.

3.3.3 Model magnetic fields

The model fields used in this exercise are the cylindrical magnetic field of a long straight wire

$$\mathbf{B}_{\mathrm{curved}}(\mathbf{r}) \;=\; B_* \left(\frac{R*}{r}\right)^2 \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \;,$$

a radial field in two dimensions,

$$\mathbf{B}_{\mathrm{rad2d}}(\mathbf{r}) \;=\; -B_* \left(\frac{R*}{r}\right)^2 \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \;,$$

a radial field in three dimensions,

$$\mathbf{B}_{\mathrm{rad3d}}(\mathbf{r}) \;=\; -B_* \left(\frac{R*}{r}\right)^3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;,$$

and a dipole field

$$\mathbf{B}_{\mathrm{dip}}(\mathbf{r}) \;=\; B_* \left(\frac{R*}{r}\right)^5 \begin{pmatrix} 3xz \\ 3yz \\ 3z^2 - r^2 \end{pmatrix} \;.$$

All fields are divergence-free and curl-free everywhere outside the origin ($\mathbf{r} = 0$).

3.3.4 Model spacecraft configuration and error analysis

In order to simplify the error analysis, we choose the spacecraft configuration sketched in figure 4 with $\Delta x_1 = \Delta x_2 = \Delta x_3 = \Delta$. The parameter R for this configuration is $R = (3/4)\Delta$. To first order in Δ/r, the spatial derivatives of the model fields estimated by finite differencing are found to be (after some lengthy but straightforward calculation)

$$\nabla \cdot \mathbf{B} \propto \frac{\Delta}{r^3} \quad , \quad (\nabla \times \mathbf{B})_z \propto \frac{\Delta}{r^3} \, ,$$

and

$$\frac{\partial B_y}{\partial x} \propto \frac{1}{r^2} \quad , \quad \frac{\partial B_x}{\partial y} \propto \frac{1}{r^2} \quad , \quad \frac{\partial B_y}{\partial y} \propto \frac{\Delta}{r^3}$$

for the first model field $\mathbf{B}_{\rm curved}$,

$$\nabla \cdot \mathbf{B} \propto \frac{\Delta}{r^3} \quad , \quad (\nabla \times \mathbf{B})_z \propto \frac{\Delta}{r^3} \, ,$$

and

$$\frac{\partial B_x}{\partial x} \propto \frac{1}{r^2} \quad , \quad \frac{\partial B_x}{\partial y} \propto \frac{\Delta}{r^3} \quad , \quad \frac{\partial B_y}{\partial y} \propto \frac{1}{r^2}$$

for the second model field $\mathbf{B}_{\rm rad2d}$, and

$$\nabla \cdot \mathbf{B} \propto \frac{\Delta}{r^4} \quad , \quad (\nabla \times \mathbf{B})_y \propto \frac{\Delta}{r^4} \quad , \quad (\nabla \times \mathbf{B})_z \propto \frac{\Delta}{r^4}$$

as well as

$$\frac{\partial B_x}{\partial x} \propto \frac{1}{r^3} \quad , \quad \frac{\partial B_y}{\partial y} \propto \frac{1}{r^3} \quad , \quad \frac{\partial B_z}{\partial z} \propto \frac{1}{r^3} \quad , \quad \frac{\partial B_x}{\partial y} \propto \frac{\Delta}{r^4} \quad , \quad \frac{\partial B_x}{\partial z} \propto \frac{\Delta}{r^4}$$

for the third model field $\mathbf{B}_{\rm rad3d}$.