Using kinetic theory to improve fluid equations for fully ionized gases

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Transport equations should

- describe the physics as faithfully as possible
- be as simple as possible

So we need to make a trade-off!

Demands of the solar wind

Basic properties, such as the mass flux and elemental composition set by processes deep in the solar atmosphere (chromosphere and transition region).

Hence solar wind models need to describe

- a vast range from neutral density $\sim 10^{20}$ m^{-3} in the chromosphere to electron/proton density $\sim 10^{6}$ m^{-3} at Earth
- temperatures from 10^4 K to more than 10^8 K (heavy ions)
- collision-dominated neutral gas and fully ionized, magnetized, collisionless gas — location of transition not known in advance
- ionization and recombination processes
- radiation
- thermal forces and heat conduction in the transition region between chromosphere and corona (balance between heat conduction and radiation sets coronal density)

$\Rightarrow$ The (fluid) model *must* be kept as simple as possible!

Less is more

Simple transport equations

- are (of course) easier to implement in a numerical model
- make the code easier to debug
- make the code faster to execute
- means that you can afford to include other processes (ionization, recombination, radiative loss, adaptive grid, ...)
- make implicit time integration schemes much easier to implement (they require lots of derivatives)
- make it easier to understand the results

Formal derivation of fluid equations

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_r f + \frac{1}{m} \mathbf{F} \cdot \nabla_v f = \left(\frac{\delta f}{\delta t} \right)_{\rm coll}$$

Multiply by some velocity power $v_i v_j \ldots$ and integrate over $\mathbf{v}$:

$$\int d\mathbf{v}\, v_i v_j \ldots \left(\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_r f + \frac{1}{m} \mathbf{F} \cdot \nabla_v f \right) = \int d\mathbf{v}\, v_i v_j \ldots \left(\frac{\delta f}{\delta t} \right)_{\rm coll}$$

- Yields infinite set of coupled moment equations that need to be closed.
- Particle, momentum, and energy conservation automatically taken care of.

Closure: Assuming a form for $f(\mathbf{r}, \mathbf{v})$

Describing a collision-dominated gas, we must insist that f is close to a Maxwellian. The "classical" Chapman-Enskog (Schunk. . .) derivation is therefore based on

$$f = f^M(1 + \phi(\mathbf{v}))$$

where

$$f^M = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{mc^2}{2kT}\right]$$

where $\mathbf{c} = \mathbf{v} - \mathbf{u}(\mathbf{r})$ and

$$\phi \propto \mathbf{a} \cdot (\mathbf{v} - \mathbf{u}) + \mathcal{O}(c^2, c^3, \cdots)$$

and $|\phi(\mathbf{v})| \ll 1$.
ϕ accounts for heat flow.

The 8-moment approximation

$$\phi(\mathbf{c}) = -\frac{m}{kTP}\mathbf{q}\cdot\mathbf{c}\left(1 - \frac{mc^2}{5kT}\right)$$

(e.g., Schunk 1977).

- Basically a first order Taylor expansion in

$$\frac{c}{\sqrt{kT/m}} \ll 1$$

- c^3 term not a free parameter but needed to assure that $\langle\mathbf{c}\rangle = 0$ so that $\mathbf{u}$ is the mean flow speed.

Linear ϕ gives poor description of Coulomb collisions

Coulomb cross section $\propto |\mathbf{v} - \mathbf{v}'|^{-4} \Rightarrow$

- Core of $f(\mathbf{v})$ should have very small departure from Maxwellian,
- which is not satisfied by $\phi \propto c$.
- For a Lorentz gas (positive ion charge $Ze \rightarrow \infty$) it can be shown that $\phi \propto c^4$ for small c.
- Electrons in plasma with weak ∇T has $\phi \propto c^3 - c^4$ (Spitzer & Härm, 1953).
- Collisional forces sensitive to core of $f(\mathbf{v})$ so errors will be large:
  - Electron heat conduction a factor 2.5 too small compared to Spitzer & Härm and Braginskii (1965).
  - Thermal force between electrons and protons (for given $\mathbf{q}$) 2.7 times larger than the Braginskii value.
  - Thermal force on α-particles (from electrons and protons) 2.6 times larger than kinetic calculation of Roussel-Dupré (1981).

The new ansatz for $f(\mathbf{v})$

Instead of (e.g, Schunk, 1977)

$$f = f_0 \left[1 - \frac{m}{kTP} \left(1 - \frac{mc^2}{5kT} \right) \mathbf{q} \cdot \mathbf{c} \right]$$

we basically choose $\phi \propto c^3$ (Killie et al., ApJ, **604**, 842, 2004):

$$f = f_0 \left[1 - \frac{m^2 c^2}{5k^2 T^2 P} \left(1 - \frac{mc^2}{7kT} \right) \mathbf{q} \cdot \mathbf{c} \right]$$

where $\mathbf{c} \equiv \mathbf{v} - \mathbf{u}$.
$\Rightarrow$ We start the Taylor expansion at third order — mathematically "incorrect" but physically much better!

The improved 8-moment approximation

Taking moments of the Boltzmann equation with the new ansatz we find ($D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$)

$$
\begin{aligned}
\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) &= \left(\frac{\delta n}{\delta t}\right)_{\mathrm{coll}} \\
nm\frac{D\mathbf{u}}{Dt} + \nabla P - nm\mathbf{G} - ne\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) &= \left(\frac{\delta \mathbf{M}}{\delta t}\right)_{\mathrm{coll}} \\
\frac{D}{Dt}\left(\frac{3}{2}P\right) + \frac{5}{2}P\left(\nabla \cdot \mathbf{u}\right) + \nabla \cdot \mathbf{q} &= \left(\frac{\delta E}{\delta t}\right)_{\mathrm{coll}} \\
\frac{D\mathbf{q}}{Dt} + \frac{7}{5}(\mathbf{q} \cdot \nabla)\mathbf{u} + \frac{7}{5}\mathbf{q}(\nabla \cdot \mathbf{u}) + \frac{2}{5}(\nabla \mathbf{u}) \cdot \mathbf{q} & \\
+\frac{5}{2}\frac{kP}{m}\nabla T - \frac{e}{m}\mathbf{q} \times \mathbf{B} &= \left(\frac{\delta \mathbf{q}'}{\delta t}\right)_{\mathrm{coll}} .
\end{aligned}
$$

The left-hand sides are identical to the “original” 8-moment approximation. But the collision terms are not. . .

The new collision terms

When $|\mathbf{u}_s - \mathbf{u}_t|/\sqrt{2kT/m} \ll 1$ and $|T_s - T_t|/T_s \ll 1$ the new momentum collision term is

$$\frac{\delta \mathbf{M}_s}{\delta t} = -\sum_t n_s m_s \nu_{st}(\mathbf{u}_s - \mathbf{u}_t) + \sum_t \nu_{st} \frac{3}{5}\frac{\mu_{st}}{kT_{st}}\left[\mathbf{q}_s\left(1 - \frac{5}{7}\frac{m_t}{m_s + m_t}\right) - \mathbf{q}_t \frac{\rho_s}{\rho_t}\left(1 - \frac{5}{7}\frac{m_s}{m_s + m_t}\right)\right]$$

compared with the “old” collision term (Schunk 1977)

$$\frac{\delta \mathbf{M}_s}{\delta t} = -\sum_t n_s m_s \nu_{st}(\mathbf{u}_s - \mathbf{u}_t) + \sum_t \nu_{st} \frac{3}{5}\frac{\mu_{st}}{kT_{st}}\left(\mathbf{q}_s - \frac{\rho_s}{\rho_t}\mathbf{q}_t\right)$$

$\Rightarrow$ Thermal force on protons from electrons reduced by factor $2/7$.

Heat flux collision term

The new collision term:

$$\frac{\delta \mathbf{q}_s}{\delta t} = -\sum_{t\neq s} \nu_{st} \left\{ E_{st}^{(1)} \mathbf{q}_s - E_{st}^{(4)} \frac{\rho_s}{\rho_t} \mathbf{q}_t + \frac{5}{2} P_s (\mathbf{u}_s - \mathbf{u}_t) \left[1 - \frac{3}{5} \frac{m_t}{m_s + m_t} \right] \right\} - \frac{16}{35} \nu_{ss} \mathbf{q}_s ,$$

Compared with the old collision term:

$$\frac{\delta \mathbf{q}_s}{\delta t} = -\sum_{t\neq s} \nu_{st} \left\{ D_{st}^{(1)} \mathbf{q}_s - D_{st}^{(4)} \frac{\rho_s}{\rho_t} \mathbf{q}_t + \frac{5}{2} P_s (\mathbf{u}_s - \mathbf{u}_t) \left[1 - \frac{3}{5} \frac{m_t}{m_s + m_t} \right] \right\} - \frac{4}{5} \nu_{ss} \mathbf{q}_s ,$$

$\Rightarrow$ For given ∇T the electron heat flux increases by factor $7/4$.

Heat flux (contd.)

The new mass factors are

$$E_{st}^{(1)} = \frac{1}{(m_s+m_t)^3}\left(3m_s^3 - \frac{1}{2}m_s^2 m_t - \frac{2}{5}m_s m_t^2 - \frac{4}{35}m_t^3\right)$$

$$E_{st}^{(4)} = \frac{1}{(m_s+m_t)^3}\left(\frac{6}{5}m_t^3 - \frac{171}{70}m_t^2 m_s - \frac{3}{7}m_t m_s^2\right).$$

and the old:

$$D_{st}^{(1)} = \frac{1}{(m_s+m_t)^2}\left(3m_s^2 + \frac{1}{10}m_s m_t - \frac{1}{5}m_t^2\right)$$

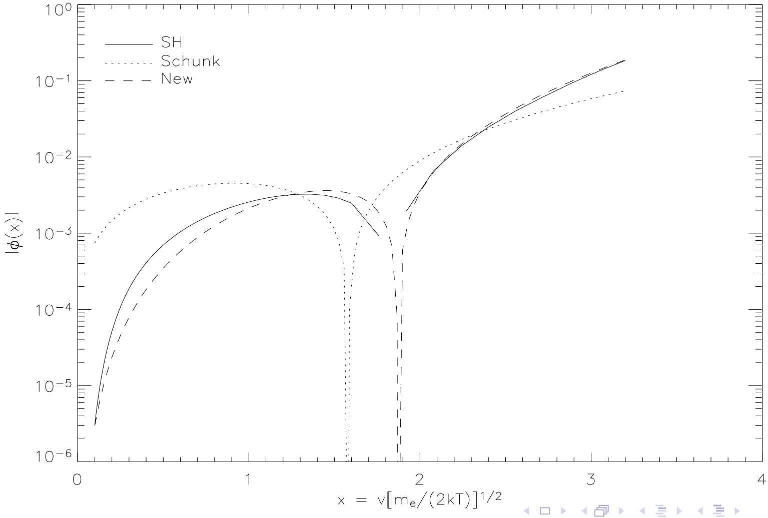
$$D_{st}^{(4)} = \frac{1}{(m_s+m_t)^2}\left(\frac{6}{5}m_t^2 - \frac{3}{2}m_s m_t\right).$$

Comparison with classical transport theory

	κ_e	$F_{e,q}$	$F_{e,\nabla T}$	$F_{\alpha,q}$	$F_{\alpha,\nabla T}$
Spitzer & Härm 1953	1				
Braginskii 1965		1	1		
Roussel-Dupré 1981				1	1
New	1.2	0.76	0.94	0.66	0.82
Schunk 1977	0.42	2.7	1.1	2.6	1.1

Departure from Maxwellian - electrons in collision-dominated plasma

$n = 10^{14}\ \mathrm{m}^{-3},\ T = 5 \times 10^{5}\ \mathrm{K},\ q = 10\ \mathrm{W/m}^2$



Extension to gyrotropic equations

Expanding instead about a bi-Maxwellian,

$$f = f^{bM}(1 + \phi)$$

with

$$f^{bM} = n(2\pi)^{-3/2}\frac{m}{kT_\perp}\sqrt{\frac{m}{kT_\parallel}}\exp\left(-\frac{m(\mathbf{v}_\perp - \mathbf{u}_\perp)^2}{2kT_\perp} - \frac{m(\mathbf{v}_\parallel - \mathbf{u}_\parallel)^2}{2kT_\parallel}\right)$$

allows description of collisionless flow of magnetized gases.

The 16-moment set

The gyrotropic equations of Demars & Schunk (J. Phys. D Appl. Phys., **12**, 1051, 1979)

- seems to give a good description of collisionless flow
  - In a weakening magnetic field $T_\perp$ is converted into $T_\|$ and increased flow speed, in agreement with the collisionless Boltzmann equation (magnetic moment conservation)
  - Comparison with kinetic model of the polar wind shows that gyrotropic equations reproduce approximately temperature anisotropy in the trans- and supersonic regions, heat fluxes show poorer agreement, and the shape of $f(\mathbf{v})$ agrees only approximately (Lie-Svendsen and Olsen, 1998).
- $\phi \propto c$ still, so heat fluxes and thermal forces do not agree with classical transport theory

Can we improve the description of collisional effects, while retaining the good description of collisionless flow?

Improved gyrotropic transport equations

(Janse et al., J. Plasma Phys., **71**, 611, 2005)

Demars & Schunk (1979) assumed (omitting stress tensors)

$$\begin{aligned}\phi &= -\frac{\beta_\perp^2}{\rho}\left(1-\frac{\beta_\perp c_\perp^2}{4}\right)\mathbf{q}^\perp\cdot\mathbf{c}_\perp - \frac{\beta_\perp\beta_\parallel}{\rho}\left(1-\frac{\beta_\perp c_\perp^2}{2}\right)\mathbf{q}^\perp\cdot\mathbf{c}_\parallel \\ &\quad -\frac{\beta_\parallel^2}{2\rho}\left(1-\frac{\beta_\parallel c_\parallel^2}{3}\right)\mathbf{q}^\parallel\cdot\mathbf{c}_\parallel - \frac{\beta_\perp\beta_\parallel}{2\rho}\left(1-\beta_\parallel c_\parallel^2\right)\mathbf{q}^\parallel\cdot\mathbf{c}_\perp\end{aligned}$$

where $\beta_{\parallel(\perp)} = m/kT_{\parallel(\perp)}$ and $\mathbf{c}=\mathbf{v}-\mathbf{u}$.
The new ansatz has only one heat flux vector and $\phi \propto c^3$:

$$\phi = \alpha_\perp \mathbf{q}\cdot\mathbf{c}_\perp c^2\left(1+\gamma_\perp c^2\right) + \alpha_\parallel \mathbf{q}\cdot\mathbf{c}_\parallel c^2\left(1+\gamma_\parallel c^2\right).$$

where...

$$\gamma_\perp = -\frac{m}{kT_\perp}\frac{4T_\perp^2 + T_\perp T_\parallel}{24T_\perp^2 + 8T_\perp T_\parallel + 3T_\parallel^2}$$

$$\gamma_\parallel = -\frac{m}{kT_\perp}\frac{2T_\perp^2 + 3T_\perp T_\parallel}{8T_\perp^2 + 12T_\perp T_\parallel + 15T_\parallel^2}$$

$$\alpha_\perp = -\frac{m^2}{nk^3}\frac{1}{T_\perp}\frac{24T_\perp^2 + 8T_\perp T_\parallel + 3T_\parallel^2}{96T_\perp^4 + 48T_\perp^3 T_\parallel + 4T_\perp^2 T_\parallel^2 + 24T_\perp T_\parallel^3 + 3T_\parallel^4}$$

$$\alpha_\parallel = -\frac{m^2}{nk^3}\frac{1}{T_\parallel}\frac{8T_\perp^2 + 12T_\perp T_\parallel + 15T_\parallel^2}{16T_\perp^4 + 48T_\perp^3 T_\parallel + 6T_\perp^2 T_\parallel^2 + 60T_\perp T_\parallel^3 + 45T_\parallel^4}.$$

. . . are needed so that **u** is the mean flow velocity and **q** the heat flux density.

One dimensional flow

"5-moment" approximation

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -u\frac{\partial u}{\partial r} - \frac{k}{m}\frac{\partial T_\parallel}{\partial r} - \frac{kT_\parallel}{nm}\frac{\partial n}{\partial r} - \frac{1}{A}\frac{dA}{dr}\frac{k}{m}\left(T_\parallel - T_\perp\right) + G \\
&\quad + \frac{e}{m}E + \frac{1}{mn}\frac{\delta M}{\delta t} \\
\frac{\partial T_\parallel}{\partial t} &= -u\frac{\partial T_\parallel}{\partial r} - 2T_\parallel\frac{\partial u}{\partial r} - \frac{1}{nk}\frac{\partial q_\parallel}{\partial r} - \frac{1}{A}\frac{dA}{dr}\frac{q_\parallel}{nk} + \frac{2}{A}\frac{dA}{dr}\frac{q_\perp}{nk} \\
&\quad + \frac{1}{nk}\frac{\delta E_\parallel}{\delta t} \\
\frac{\partial T_\perp}{\partial t} &= -u\frac{\partial T_\perp}{\partial r} - \frac{1}{A}\frac{dA}{dr}uT_\perp - \frac{1}{nk}\frac{\partial q_\perp}{\partial r} - \frac{2}{A}\frac{dA}{dr}\frac{q_\perp}{nk} + \frac{1}{nk}\frac{\delta E_\perp}{\delta t}
\end{aligned}
$$

are *formally* identical to the corresponding "old" (Demars & Schunk, 1979) equation set.

The heat flux equation

$$\begin{aligned}\frac{\partial q}{\partial t} &= -u\frac{\partial q}{\partial r} - 2q_{\|}\frac{\partial u}{\partial r} - \frac{1}{2}q_{s\|}u\frac{1}{A}\frac{dA}{dr} - 2q_{\perp}\frac{\partial u}{\partial r} - 2q_{\perp}u\frac{1}{A}\frac{dA}{dr} \\ &\quad -\frac{k^2 n T_{\|}}{m}\frac{\partial}{\partial r}\left(\frac{3}{2}T_{\|} + T_{\perp}\right) - \frac{1}{A}\frac{dA}{dr}\frac{k^2 n T_{\perp}}{m}\left(T_{\|} - T_{\perp}\right) \\ &\quad +\frac{\delta q'}{\delta t}\end{aligned}$$

which is also formally equal to the sum of "old" equations for $q = q_{\|}/2 + q_{\perp}$. However, with

$$\begin{aligned}q_{\|} &= 30q\frac{T_{\|}^3(4T_{\perp} + 3T_{\|})}{16T_{\perp}^4 + 48T_{\perp}^3T_{\|} + 6T_{\perp}^2T_{\|}^2 + 60T_{\perp}T_{\|}^3 + 45T_{\|}^4} \\ q_{\perp} &= 2q\frac{T_{\perp}^2(8T_{\perp}^2 + 24T_{\perp}T_{\|} + 3T_{\|}^2)}{16T_{\perp}^4 + 48T_{\perp}^3T_{\|} + 6T_{\perp}^2T_{\|}^2 + 60T_{\perp}T_{\|}^3 + 45T_{\|}^4}.\end{aligned}$$

The new gyrotropic set

- eliminates separate equations for $q_{\|}$ and $q_{\perp}$
- gives the same good description of collisionless flow (magnetic moment conservation)
- reduces to the (new) isotropic equation set when $T_{\|} = T_{\perp}$
- when temperature anisotropies are small the same collision terms may be used (saving us lots of work. . .)

A full solar wind model

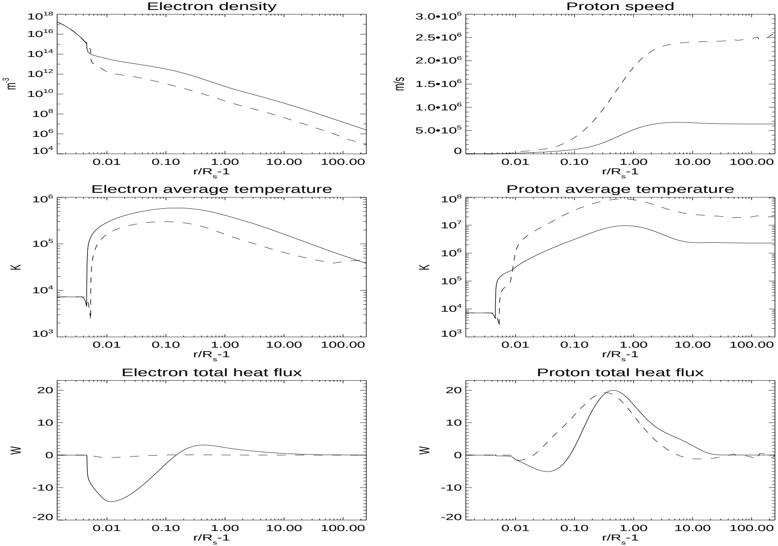
(Janse et al., Phys. Scripta, **T122**, 66, 2006)

- extending from the chromosphere to Earth
- solves the coupled equations for neutral hydrogen, protons, and electrons
- ionization of hydrogen and electron-proton recombination included
- radiative loss included (assuming optically thin loss)
- adaptive grid allowing transition region to adjust dynamically
- (semi-) implicit time integration
- "radiative energy balance" model (Withbroe 1988) — heating function and geometry only input parameters

Comparison of new and old equations

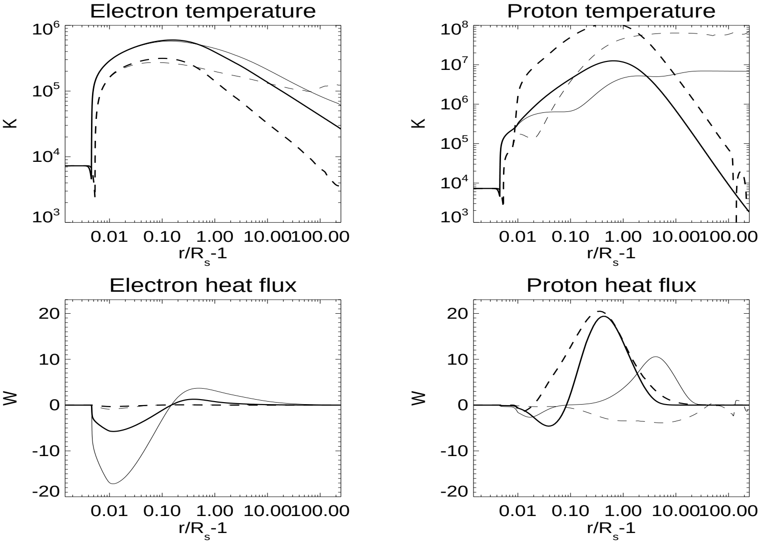
Perpendicular proton heating, rapidly expanding geometry (coronal hole)

solid: new, dashed: old



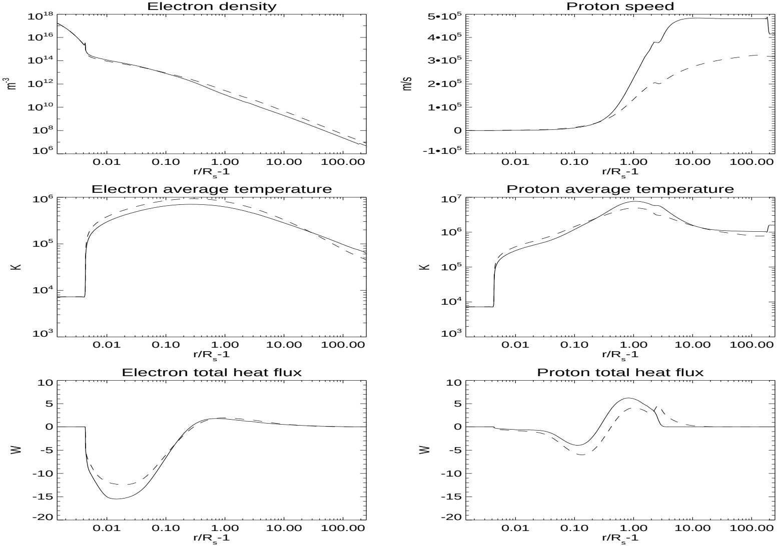
Temperature anisotropy

solid: new, dashed: old, thin: ∥, thick: ⊥



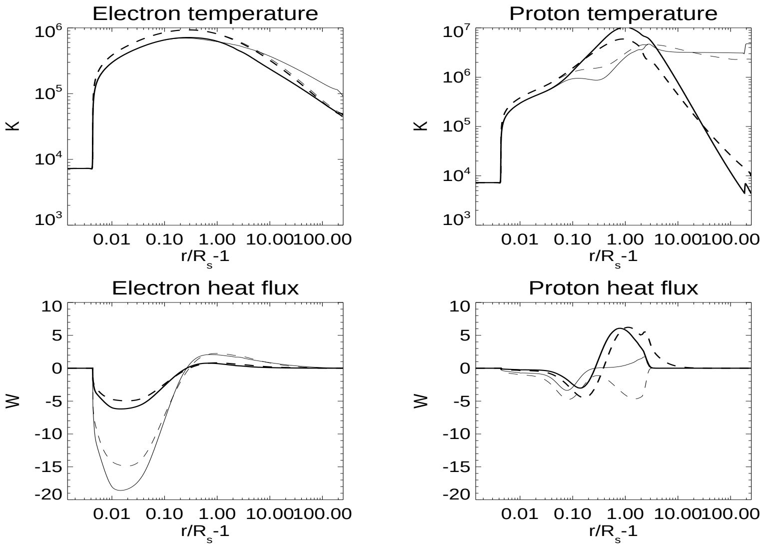
Radially expanding flow

solid: new, dashed: old



Radially expanding wind; anisotropy

solid: new, dashed: old, thin: $\|$, thick: $\perp$



- correct description of heat conduction essential!
- with new equations a reasonable high-speed wind can be obtained with only perpendicular proton heating — not so with old equation set
- geometry has much less influence on the solution with the new equations
- very large difference in solution found in a coronal hole geometry; much smaller difference in radially expanding flow
- two sets give quite similar behaviour in collisionless flow (as we hoped for):
    - $T_\perp \propto r^{-2}$
    - $T_\parallel \approx$ constant

Conclusions

- New isotropic and gyrotropic fluid equations improve description of collisions in fully ionized gases — particularly heat flow and thermal forces
- Gyrotropic equations provide a good description of collisionless flow as well (particularly magnetic moment conservation)
- Gyrotropic eqs. can accommodate parallel/perpendicular heating (e.g., cyclotron waves).
- Equations intended to be "simple" — straightforward to include in numerical model.
- Applicable to a plasma of arbitrary composition.
- Solves separate equation for heat flux; no assumption about classical/nonclassical heat flow.
- Contains the transition from collision-dominated to collisionless flow.

To do

Protons in the solar corona — transition to collisionless flow

- Collision terms derived in the semilinear approximation $|u_s - u_t| \ll \sqrt{kT/m}$ — how good are they in the corona?
- How well is the proton heat flux described?
- Do the fluid equations describe the transition from collision-dominated to collisionless proton heat flow reasonably correct?

$\Rightarrow$ Kinetic modelling of protons, including self-collisions, in the corona would be very useful!