Generalized Transport Equations and Distribution Functions for Space Plasmas

R. W. Schunk and A. R. Barakat

Center for Atmospheric and Space Sciences
Utah State University, Logan, Utah
USA

Outline

- General transport equations
- Closure of transport equations
  - Chapman-Enskog (collision-dominated)
  - Grad’s approach (Maxwellian: 5-m, 13-m, 20-m)
  - Far from Equilibrium (Bi-Maxwellian: 6-m, 16-m)
- Convergence of transport equations
- The Solar wind
- Summary

Formulations

- Hydrodynamic
- Magnetohydrodynamic
- Chapman - Enskog
- Hydromagnetic
- Generalized Transport
- Kinetic
- Semi-kinetic
- Multistream

Boltzmann Equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v}_s \cdot \nabla f_s + \left[\mathbf{G} + \frac{e_s}{m_s} \left(\mathbf{E} + \frac{1}{c} \mathbf{v}_s \times \mathbf{B} \right) \right] \cdot \nabla_{v_s} f_s = \frac{\delta f_s}{\delta t}$$

$f_s(\mathbf{r}, \mathbf{v}_s, t) =$ distribution function

$e_s =$ charge

$m_s =$ mass

$\mathbf{v}_s =$ velocity

$\frac{\partial}{\partial t} =$ time derivative

$\nabla =$ spatial gradient

$\nabla_{v_s} =$ velocity gradient

$\mathbf{E} =$ electric field

$\mathbf{B} =$ magnetic field

$\mathbf{G} =$ gravity

$c =$ speed of light

$\frac{\delta f_s}{\delta t} =$ collision operator

Transport Equations

Take velocity moments of the Boltzmann equation to get conservation equations for density, drift velocity, temperature, etc.

- **What velocity moments do we need?**
- **How do you close system of transport equations?**
- **How do you evaluate collision integrals?**

Series Expansion

$$f_s(\mathbf{r},\mathbf{v}_s,t) = f_s^{(0)}(\mathbf{r},\mathbf{v}_s,t)\sum_{\alpha} a_{\alpha}(\mathbf{r},t)M_{\alpha}(\mathbf{v}_s)$$

where

$f_s^{(0)}(\mathbf{r},\mathbf{v}_s,t) =$ zeroth-order distribution

$a_{\alpha} =$ expansion coefficient

$M_{\alpha} =$ polynomial set (orthogonal & complete)

Chapman-Enskog (1901)

- Velocity moments relative to **u**

$$n_s = \int d\mathbf{v}_s \, f_s$$

$$\mathbf{u}_s = \frac{1}{n_s} \int d\mathbf{v}_s \, f_s \mathbf{v}_s$$

$$\mathbf{u} = \frac{\sum_{\mathbf{s}} n_s m_s \mathbf{u}_s}{\sum_{\mathbf{s}} n_s m_s}$$

average velocity
of gas mixture

$$T_s^* = \frac{m_s}{3kn_s} \int d\mathbf{v}_s \, f_s \left(\mathbf{v}_s - \mathbf{u}\right)^2$$

$$\mathbf{q}_s^* = \frac{1}{2} m_s \int d\mathbf{v}_s \, f_s \left(\mathbf{v}_s - \mathbf{u}\right)^2 \left(\mathbf{v}_s - \mathbf{u}\right)$$

- $f^{(0)} =$ Maxwellian about $\mathbf{u}$
- Additional assumption of collision dominance

  $n, \mathbf{u}, T, \sim O(1)$

  $\mathbf{q}, \boldsymbol{\tau} \sim n, \mathbf{u}, T, \nabla n, \nabla \mathbf{u}, \nabla T$

- Introduced diffusion velocities

  $\mathbf{w}_S = \mathbf{u}_s - \mathbf{u}$

Grad (1949)

- Velocity moments relative to $\mathbf{u}_s$

$$n_s = \int d\mathbf{v}_s f_s$$

$$\mathbf{u}_s = \frac{1}{n_s}\int d\mathbf{v}_s f_s \mathbf{v}_s$$

$$T_s = \frac{m_s}{3kn_s}\int d\mathbf{v}_s f_s \left(\mathbf{v}_s - \mathbf{u}_s\right)^2$$

$$\mathbf{q}_s = \frac{1}{2} m_s \int d\mathbf{v}_s f_s \left(\mathbf{v}_s - \mathbf{u}_s\right)^2 \left(\mathbf{v}_s - \mathbf{u}_s\right)$$

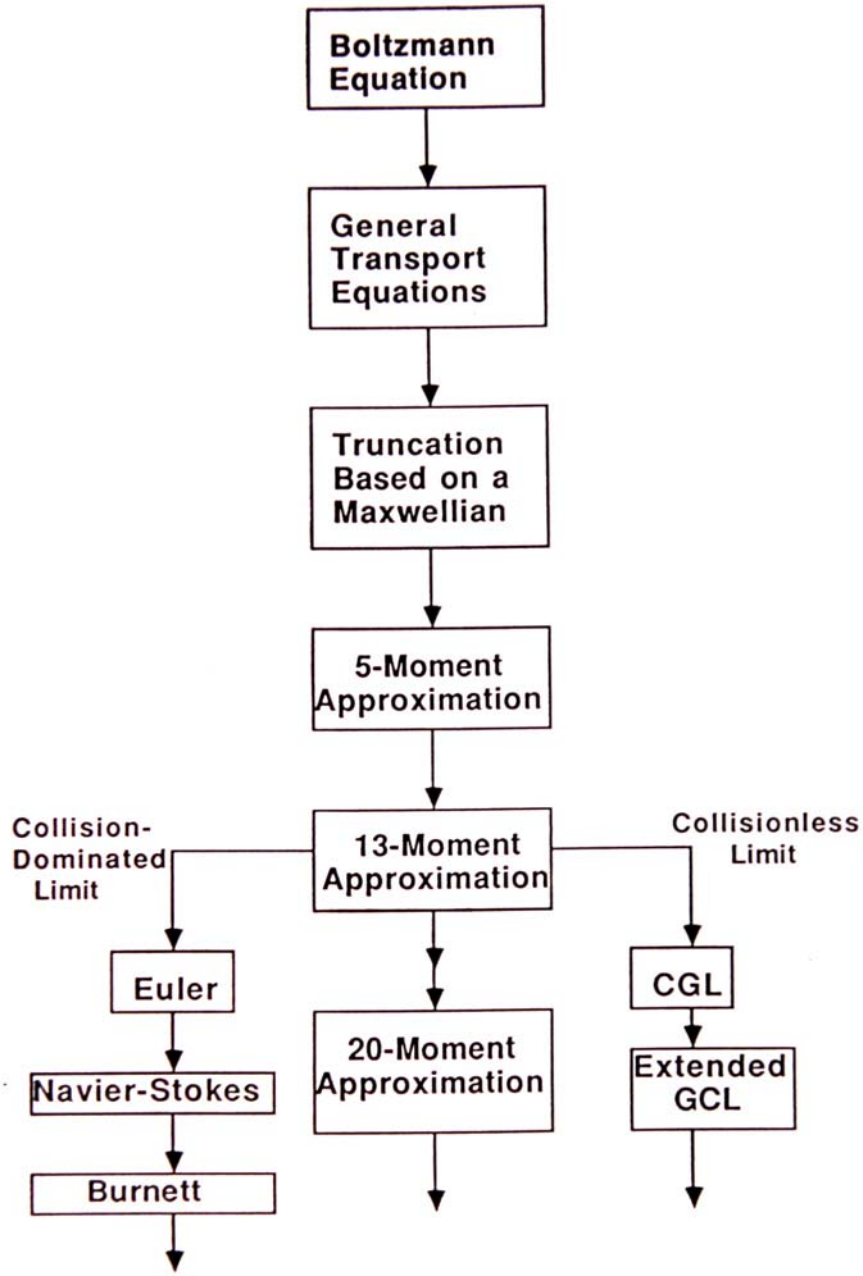
- $f^{(0)}$ = Maxwellian about $\mathbf{u}_s$
- No additional assumptions

$$n, \mathbf{u}, T, \mathbf{q}, \boldsymbol{\tau} \sim O(1)$$

Comparison

- Grad approach is better than the Chapman - Enskog approach for nonequilibrium flows.

- The Chapman-Enskog equations can be obtained from Grad's equations in the limit of a large collision frequency.


Boltzmann Equation
General Transport Equations
Truncation Based on a Maxwellian
5-Moment Approximation
13-Moment Approximation
Collision-Dominated Limit
Collisionless Limit
Euler
Navier-Stokes
Burnett
20-Moment Approximation
CGL
Extended GCL

Moments of the Boltzmann Equation

1 - Continuity Equation - n
$\mathbf{c}$ - Momentum Equation - $\mathbf{u}$
c^2 - Energy Equation - T
$\mathbf{cc}$ - Stress Tensor Equation - $\boldsymbol{\tau}$
$c^2\mathbf{c}$ - Heat Flow Equation - $\mathbf{q}$

General Transport Equations

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = \frac{\delta n_s}{\delta t}$$

$$n_s m_s \left[\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right] + \nabla p_s + \nabla \cdot \boldsymbol{\tau}_s - n_s m_s \mathbf{G}$$
$$-n_s e_s \left[\mathbf{E} + \frac{1}{c} \mathbf{u}_s \times \mathbf{B} \right] = \frac{\delta \mathbf{M}_s}{\delta t}$$

$$\left[\frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla \right] \left(\frac{3}{2} p_s \right) + \frac{5}{2} p_s (\nabla \cdot \mathbf{u}_s) + \nabla \cdot \mathbf{q}_s + \boldsymbol{\tau}_s : \nabla \mathbf{u}_s = \frac{\delta \mathbf{E}_s}{\delta t}$$

Stress Tensor Equation

Heat Flow Equation

.

.

….etc.

Closure of Transport Equations

$$f_s(\mathbf{r},\mathbf{v}_s,t) = f_s^{(M)}\sum_{\alpha} a_\alpha(\mathbf{r},t)M_\alpha(\mathbf{c}_s)$$

$\mathbf{c}_s = \mathbf{v}_s - \mathbf{u}_s$ random velocity

$f_s \approx f_s^{(M)}$ 5-moment

$$f_s \approx f_s^{(M)}\left[1+\left(...\right)\boldsymbol{\tau}_s : \mathbf{c}_s\mathbf{c}_s + \left(...\right)\mathbf{q}_s.\mathbf{c}_s\right]$$

13-moment

$$f_s \approx f_s^{(M)}\left[1+\left(...\right)\boldsymbol{\tau}_s : \mathbf{c}_s\mathbf{c}_s + \mathbf{Q}_s : \mathbf{c}_s\mathbf{c}_s\mathbf{c}_s\right]$$

20-moment

.

.

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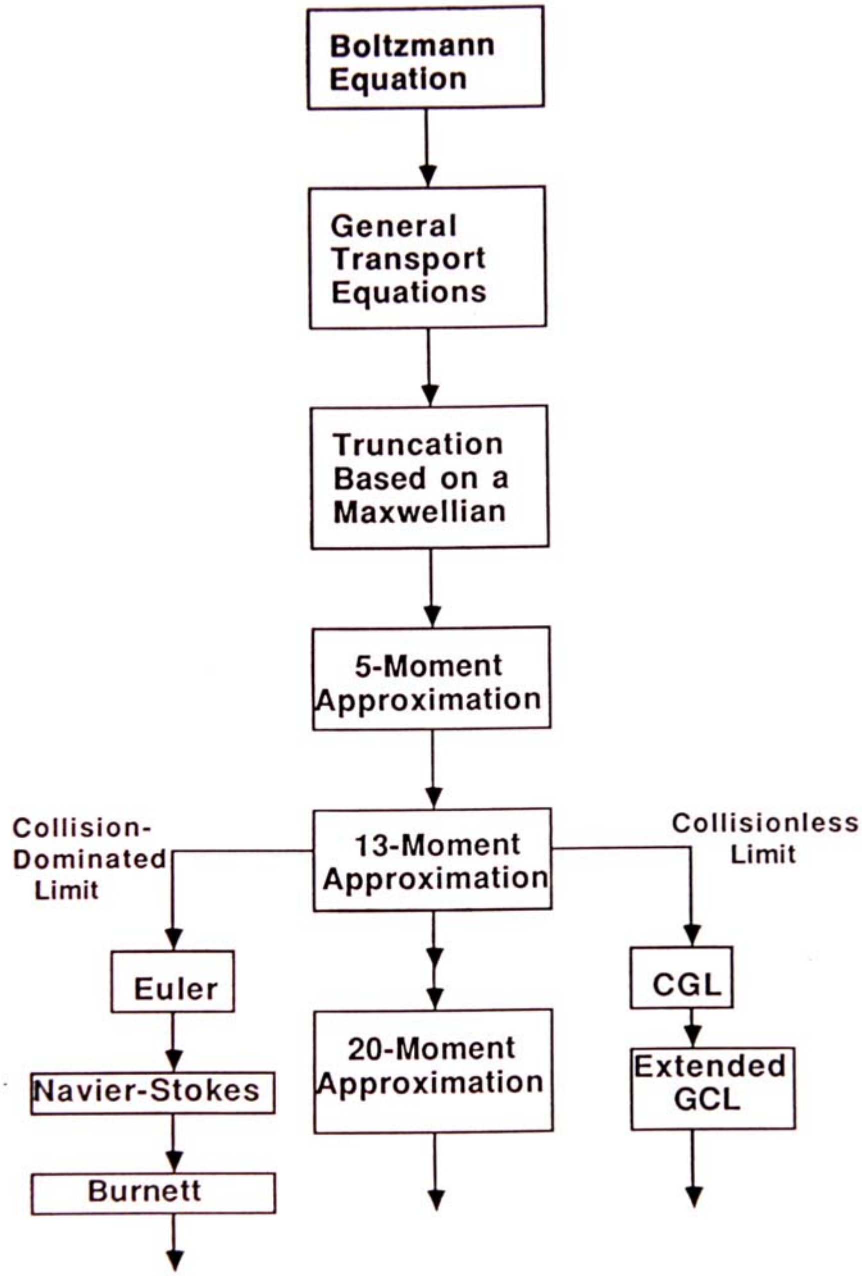
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Expansion of the Distribution Function

(*a*) 5 - MOMENT APPROXIMATION

Maxwellian Distribution

n - 1 $\mathbf{q}$ = 0
$\mathbf{u}$ - 3 $\boldsymbol{\tau}$ = $\underline{0}$
T - 1 5

(*b*) 13 - MOMENT APPROXIMATION

Maxwellian Distribution

n - 1 $\mathbf{q}$ = 3
$\mathbf{u}$ - 3 $\boldsymbol{\tau}$ = $\underline{5}$
T - 1 13

(*c*)20 - MOMENT APPROXIMATION

Maxwellian Distribution

n - 1 $\mathbf{Q}$ = 10
$\mathbf{u}$ - 3 $\boldsymbol{\tau}$ = $\underline{5}$
T - 1 20

13-Moment Approximation

Maxwellian with heat flow and stress corrections

Processes→ Ordinary Diffusion

Thermal Diffusion
Thermal Conduction
Diffusion - Thermal Heat Flow
Thermoelectric Effects
Viscosity
Collisionless Heat Flow
Collisionless Viscosity
Temperature Anisotropies

Collision-Dominated Gas

$$n, \mathbf{u}, T \sim 1$$

$$\mathrm{q}, \tau \sim 1/\nu$$

$O(1)$ → Euler

$O(1/\nu)$ → Navier-Stokes

$O(1/\nu^2)$ → Extended Navier-Stokes

Collision-Dominated Gas

1. Continuity Equation - n_s
2. Momentum Equation - $\mathbf{u}_s$
3. Energy Equation - T_s

(*a*) *O*(1) → Euler

$$\boldsymbol{\tau}_s = 0$$

$$\mathbf{q}_s = 0$$

(*b*) *O*(1/ν) → Navier-Stokes

$$\boldsymbol{\tau}_s = -\eta_s \left[\nabla \mathbf{u}_s + (\nabla \mathbf{u}_s)^T - \frac{2}{3} (\nabla \bullet \mathbf{u}_s) \mathbf{I} \right]$$

$$\mathbf{q}_s = -\lambda_{st} \nabla T_s - \lambda_{st}^{'} \nabla T_t + R_{st} (\mathbf{u}_s - \mathbf{u}_t)$$

(*c*) *O*(1/ν2) → Exteneded Navier-Stokes

Additional corrections

Collision - Dominated Gas

- Grad Method
- Burgers' Linear Collision Terms
- Navier-Stokes Limit

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) = 0$$

$$n_s m_s \left[\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s \right] + \nabla p_s + \nabla \cdot \boldsymbol{\tau}_s - n_s m_s \mathbf{G} \text{ - } n_s e_s \left[\mathbf{E} + \frac{1}{c} \mathbf{u}_s \times \mathbf{B} \right] =$$

$$\sum_t n_s m_s \nu_{st} (\mathbf{u}_t - \mathbf{u}_s) + \sum_t \nu_{st} \frac{z_{st} \mu_{st}}{k T_{st}} (\mathbf{q}_s - \frac{\rho_s}{\rho_t} \mathbf{q}_t)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla \right] \left(\frac{3}{2} p_s \right) + \frac{5}{2} p_s \left(\nabla \cdot \mathbf{u}_s \right)^T + \nabla \cdot \mathbf{q}_s + \boldsymbol{\tau}_s : \nabla \mathbf{u}_s = \frac{\delta E_s}{\delta t}$$

$$\boldsymbol{\tau}_s = -\eta_s \left[\nabla \mathbf{u}_s + \left(\nabla \mathbf{u}_s \right)^T - \frac{2}{3} (\nabla \bullet \mathbf{u}_s) \mathbf{I} \right]$$

$$\mathbf{q}_s = -\lambda_{st} \nabla T_s - -\lambda_{st}^{'} \nabla T_s + R_{st} (\mathbf{u}_s - \mathbf{u}_t)$$

Collisionless Plasma in a Strong **B**-Field

$n, \mathbf{u}, T_{\|}, T_{\perp} \sim 1$ (Bi-Maxwellian)

$\mathbf{q}, \boldsymbol{\tau} \sim 1/\Omega$

$O(1) \rightarrow$ Chew-Goldberger-Low Equations (CGL)

$O\,(1/\Omega) \rightarrow$ Extended CGL Equations

Far-From-Equilibrium Flows

$$f_s(\mathbf{r},\mathbf{v}_s,t) = f_s^{(0)} \sum_{\alpha} a_\alpha(\mathbf{r},t) M_\alpha(\mathbf{c}_s)$$

$$\mathbf{c}_s = \mathbf{v}_s - \mathbf{u}_s$$

$f_s^{(0)} =$ weight factor

1. - Chapman and Enskog
   - Grad
   - Burgers
   - Euler
   - Navier-Stokes

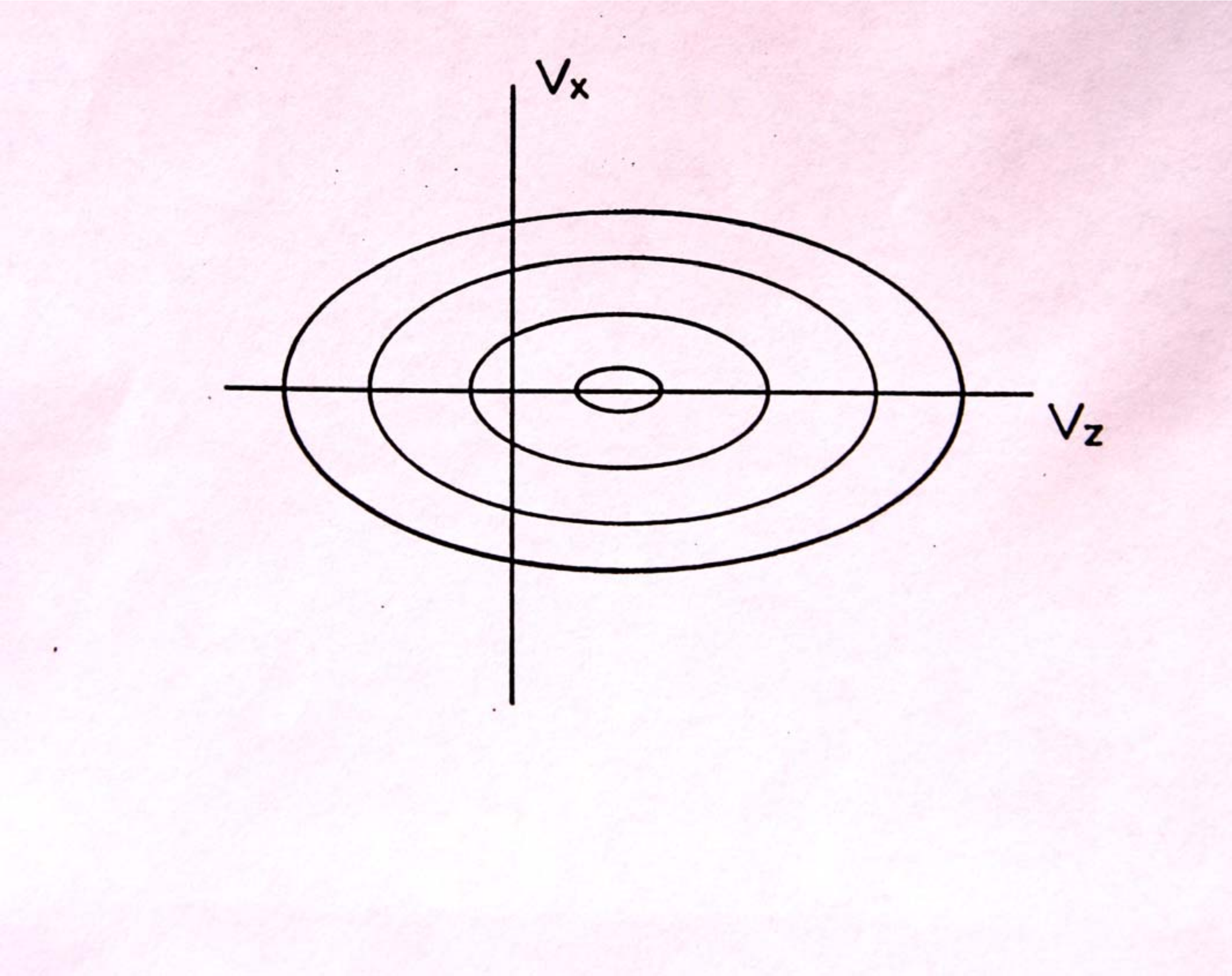
   all are based on a Maxwellian weight factor

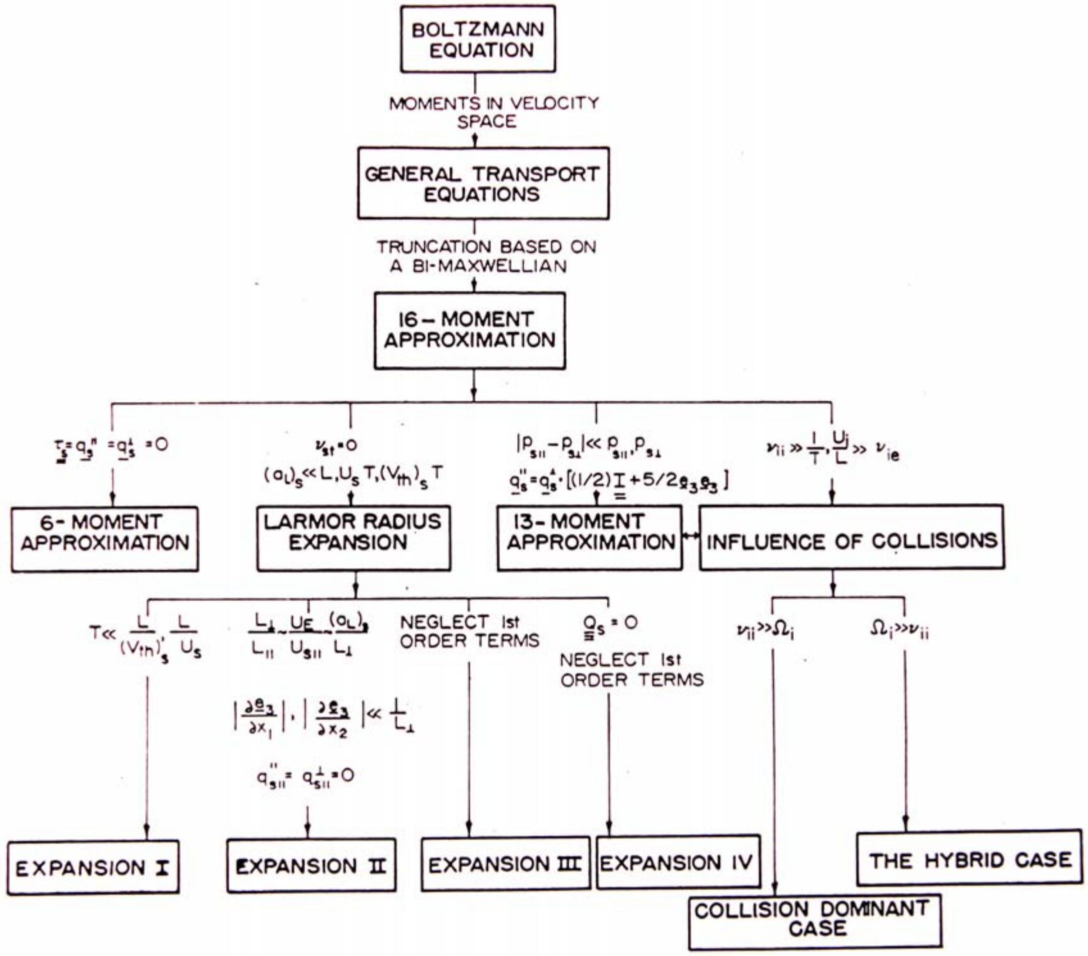
2. For far- from-equilibrium flows, the solutions converge slowly.

3. Need to select a new weight factor and generate new transport equations.

Drifting Bi-Maxwellian

$n, \mathbf{u}, T_{\parallel}, T_{\perp}$




BOLTZMANN EQUATION
MOMENTS IN VELOCITY SPACE
GENERAL TRANSPORT EQUATIONS
TRUNCATION BASED ON A BI-MAXWELLIAN
16 – MOMENT APPROXIMATION
$\underline{\underline{\tau}}_s = \underline{q}_s^{\parallel} = \underline{q}_s^{\perp} = 0$
$\nu_{st} = 0$
$(a_L)_s \ll L, U_s T, (V_{th})_s T$
$|p_{s\parallel} - p_{s\perp}| \ll p_{s\parallel}, p_{s\perp}$
$\underline{q}_s^{\parallel} = \underline{q}_s^{\perp} \cdot [(1/2)\underline{\underline{I}} + 5/2\, \mathbf{e}_3 \mathbf{e}_3]$
$\nu_{ii} \gg \frac{1}{T}, \frac{U_i}{L} \gg \nu_{ie}$
6- MOMENT APPROXIMATION
LARMOR RADIUS EXPANSION
13- MOMENT APPROXIMATION
INFLUENCE OF COLLISIONS
$T \ll \frac{L}{(V_{th})_s}, \frac{L}{U_s}$
$\frac{L_\perp}{L_\parallel} \sim \frac{U_E}{U_{s\parallel}} \sim \frac{(a_L)_s}{L_\perp}$
$\left|\frac{\partial \mathbf{e}_3}{\partial x_1}\right|, \left|\frac{\partial \mathbf{e}_3}{\partial x_2}\right| \ll \frac{1}{L_\perp}$
$q_{s\parallel}^{\parallel} = q_{s\parallel}^{\perp} = 0$
NEGLECT 1st ORDER TERMS
$\underline{\underline{Q}}_s = 0$
NEGLECT 1st ORDER TERMS
$\nu_{ii} \gg \Omega_i$
$\Omega_i \gg \nu_{ii}$
EXPANSION I
EXPANSION II
EXPANSION III
EXPANSION IV
THE HYBRID CASE
COLLISION DOMINANT CASE

Applications

- Convecting Ionosphere
- Polar Wind
- Solar Wind
- Plasmasphere

Simple Flow Situation

- Weakly-ionized plasma (*i-n* collisions)
- Homogeneous
- Steady State
- Simple Relaxation Collision Model
- Constant **E** and **B** fields

$$\frac{e_i}{m_i}\left[\mathbf{E}+\frac{1}{c}v_i\times\mathbf{B}\right]\cdot\nabla_{v_i} f_i = -\nu_i(f_i - f_i^{(M)})$$

$f_i^{(M)} =$ Maxwellian

$\nu_i =$ constant

Exact solution possible for arbitrary: $\frac{\nu_i}{\Omega_i}$, $\mathbf{E}$, $\mathbf{B}$

Convergence Study

- Grad Maxwellian-Based Expansion

  5 - m

  13 - m

  20 - m

- Bi-Maxwellian-Based Expansion

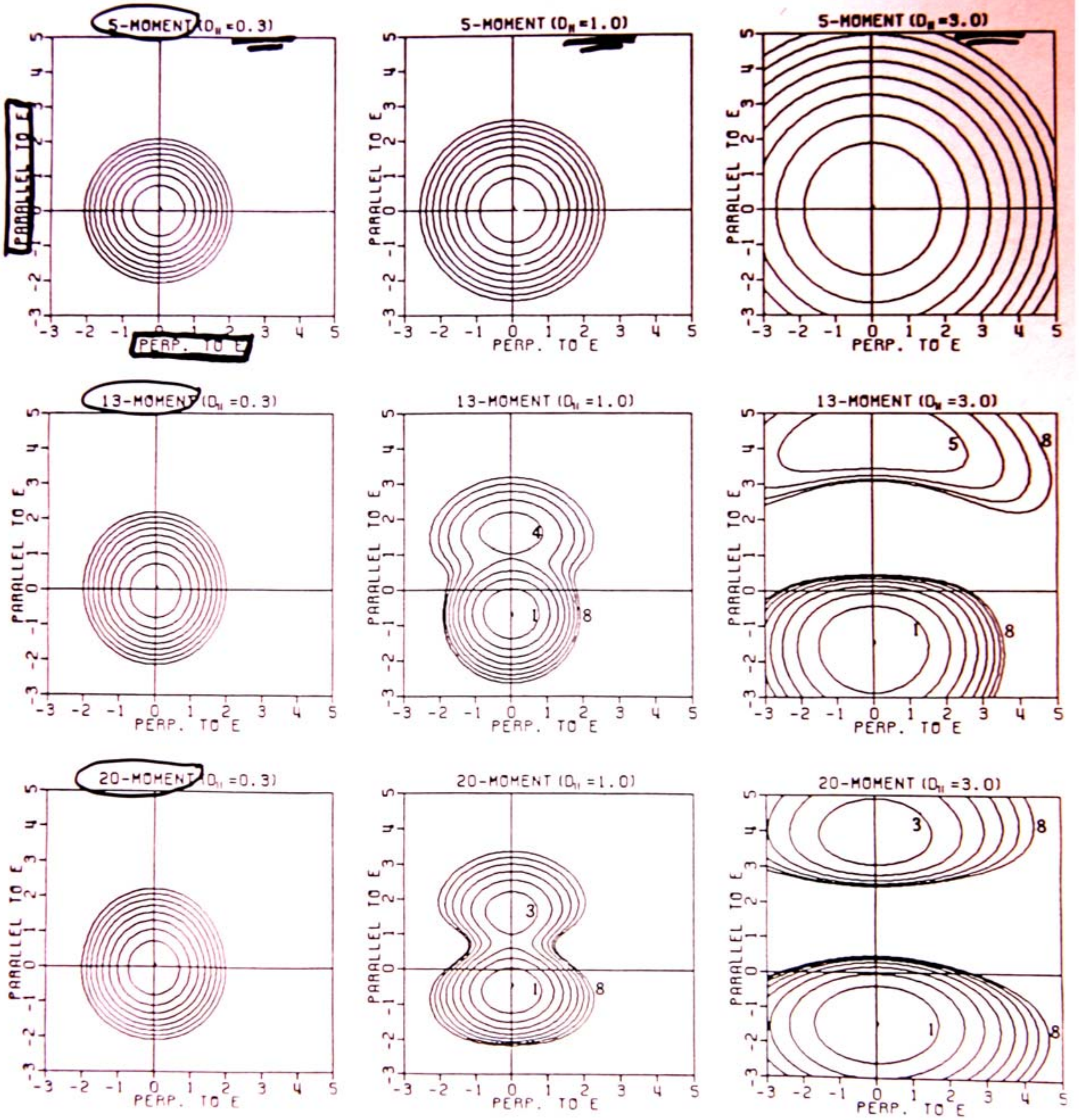
  6 - m

  16 - m

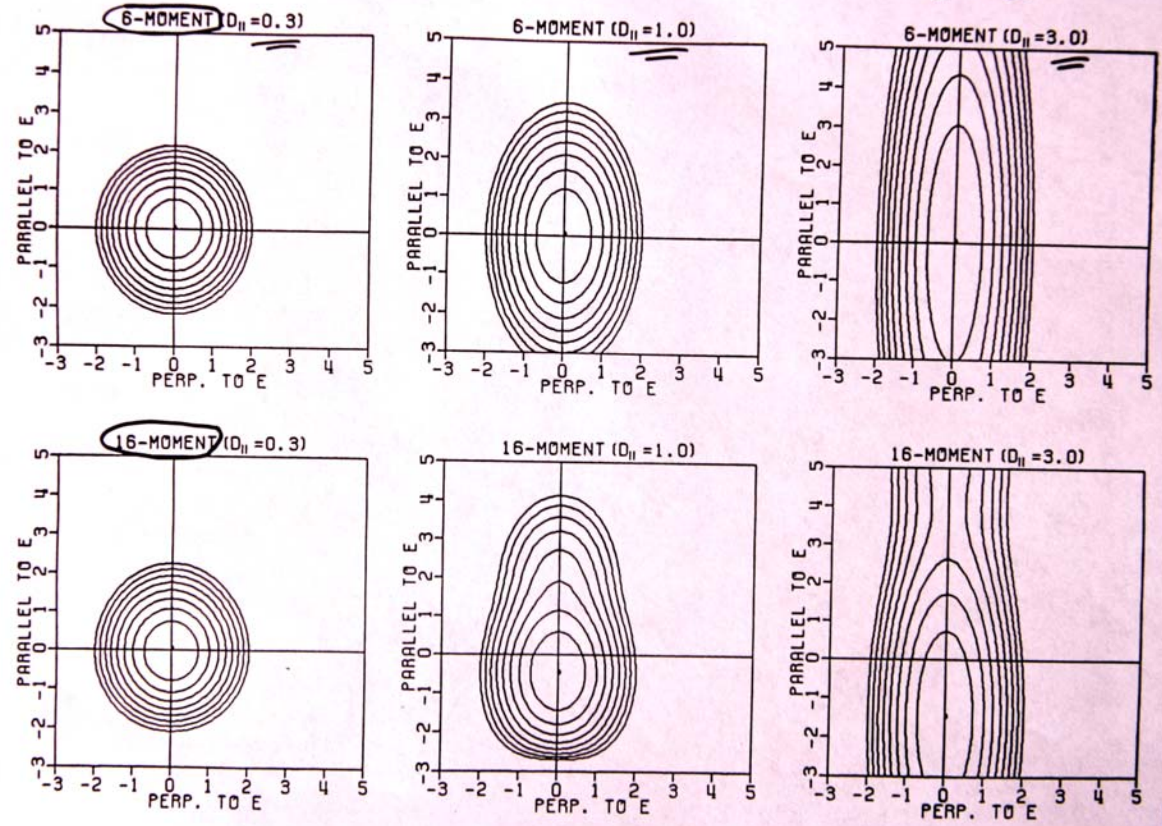
- Compare different series expansions to exact solution to test convergence of the series expansions.

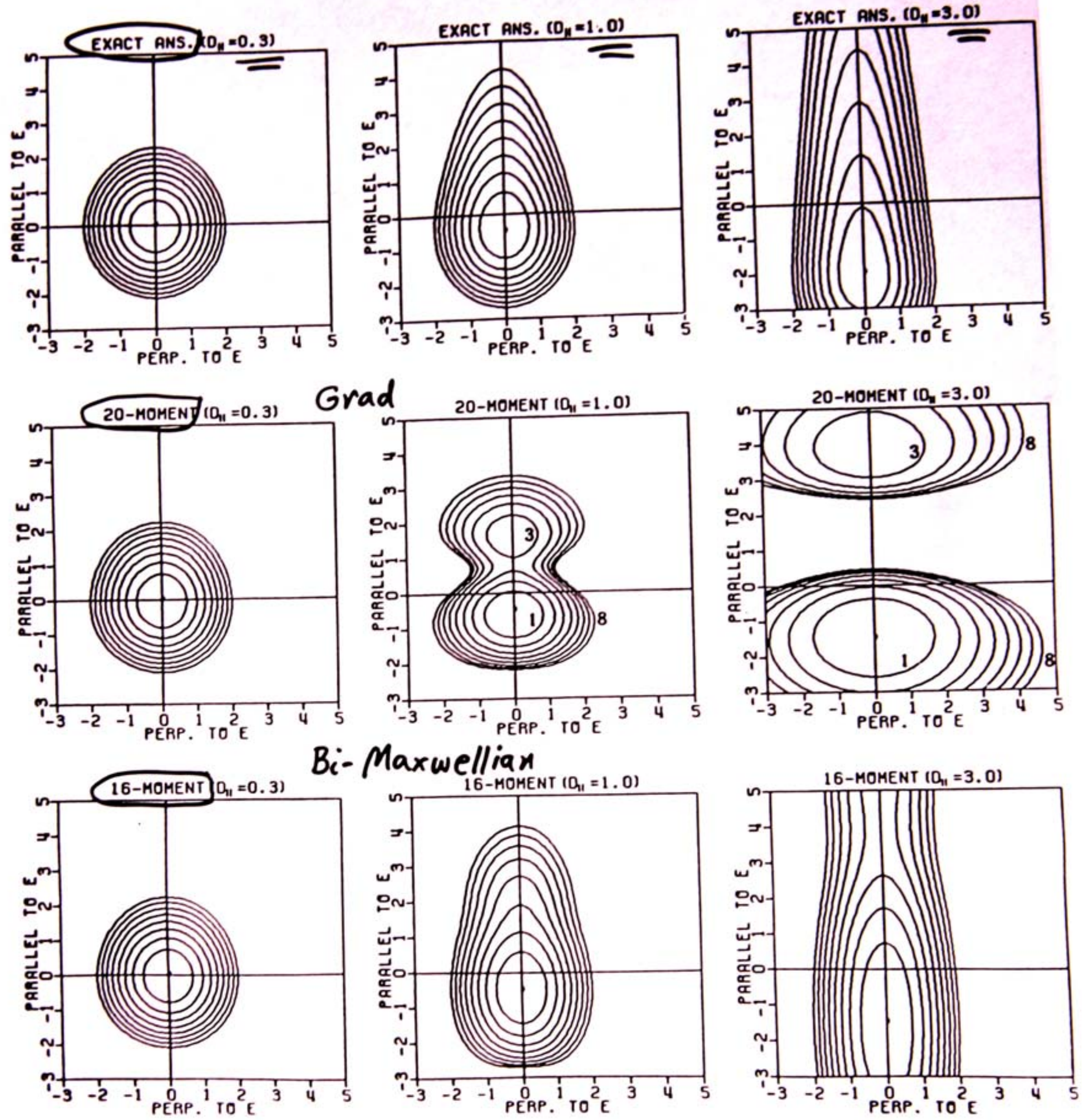
Grad Expansion

$\underline{E} \parallel \underline{B}$



Bi-Maxwellian Expansion $\underline{E} \parallel \underline{B}$




EXACT ANS. (D_{II}=0.3)
EXACT ANS. (D_{II}=1.0)
EXACT ANS. (D_{II}=3.0)
Grad
20-MOMENT (D_{II}=0.3)
20-MOMENT (D_{II}=1.0)
20-MOMENT (D_{II}=3.0)
Bi-Maxwellian
16-MOMENT (D_{II}=0.3)
16-MOMENT (D_{II}=1.0)
16-MOMENT (D_{II}=3.0)
PARALLEL TO E
PERP. TO E
-3
-2
-1
0
1
2
3
4
5
1
3
8

Solar Wind

- Standard Hydrodynamic Equations

  $n, \mathbf{u}, T_p, T_e, \mathbf{q}_p, \mathbf{q}_e$

  with spitzer conductivities
- Bi-Maxwellian based 16-Moment Equations
- B. C. at 1 A. U.

  n = 16 cm^{-3}

  u = 400 Km/s

  T_e = 100,000 K

  $$\begin{cases} T_{e\|} = 150,000\,\mathrm{K} \\ T_{e\perp} = 75,000\,\mathrm{K} \end{cases}$$

  $T_p = 40{,}000$ K

$$\begin{cases} T_{p\parallel} = 80,000 \text{ k} \\ T_{p\perp} = 20,000 \text{ k} \end{cases}$$

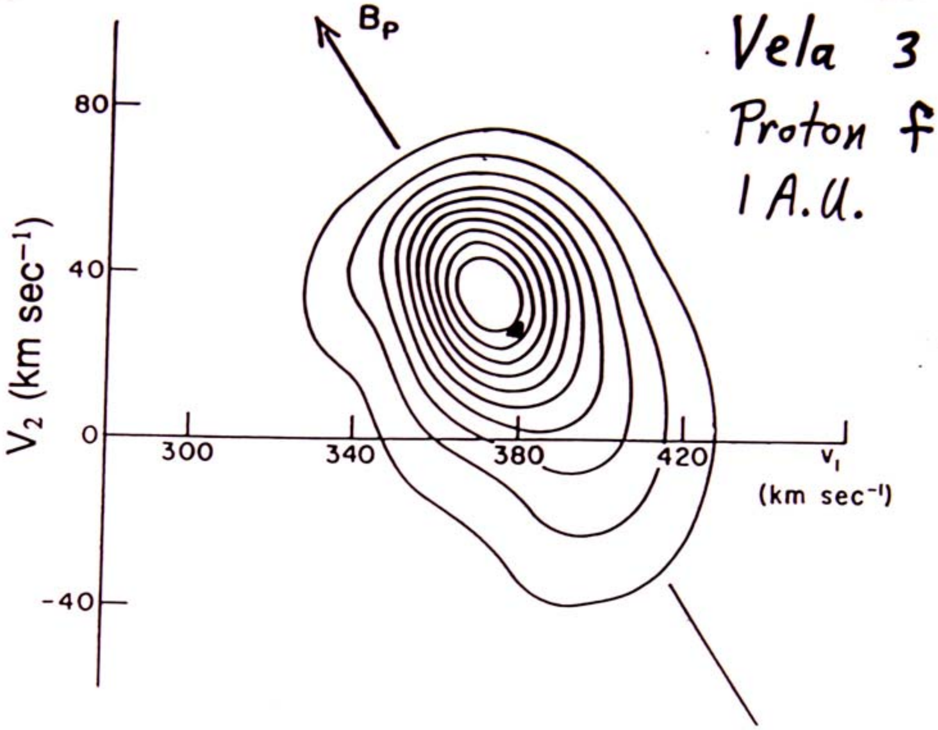
$$q_e = 0.015 \quad \mathrm{erg\ cm^{-2}\ s^{-1}}$$

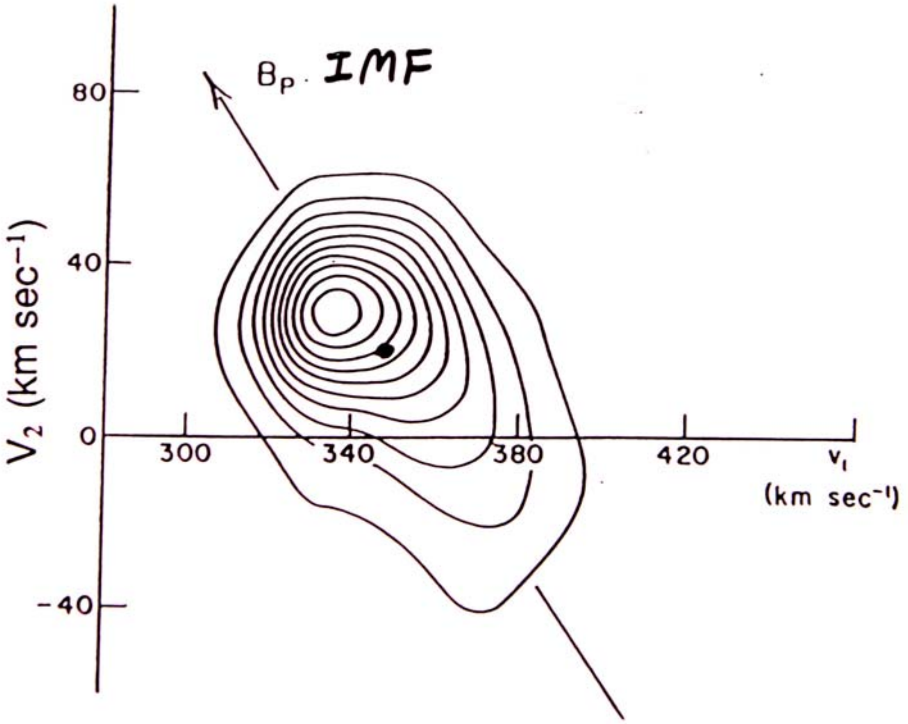
$$\begin{cases} q_p^{\parallel} = 0.01 \qquad erg\ \mathrm{cm}^{-2} s^{-1} \\ q_e^{\perp} = 0.01 \qquad erg\ \mathrm{cm}^{-2} s^{-1} \end{cases}$$

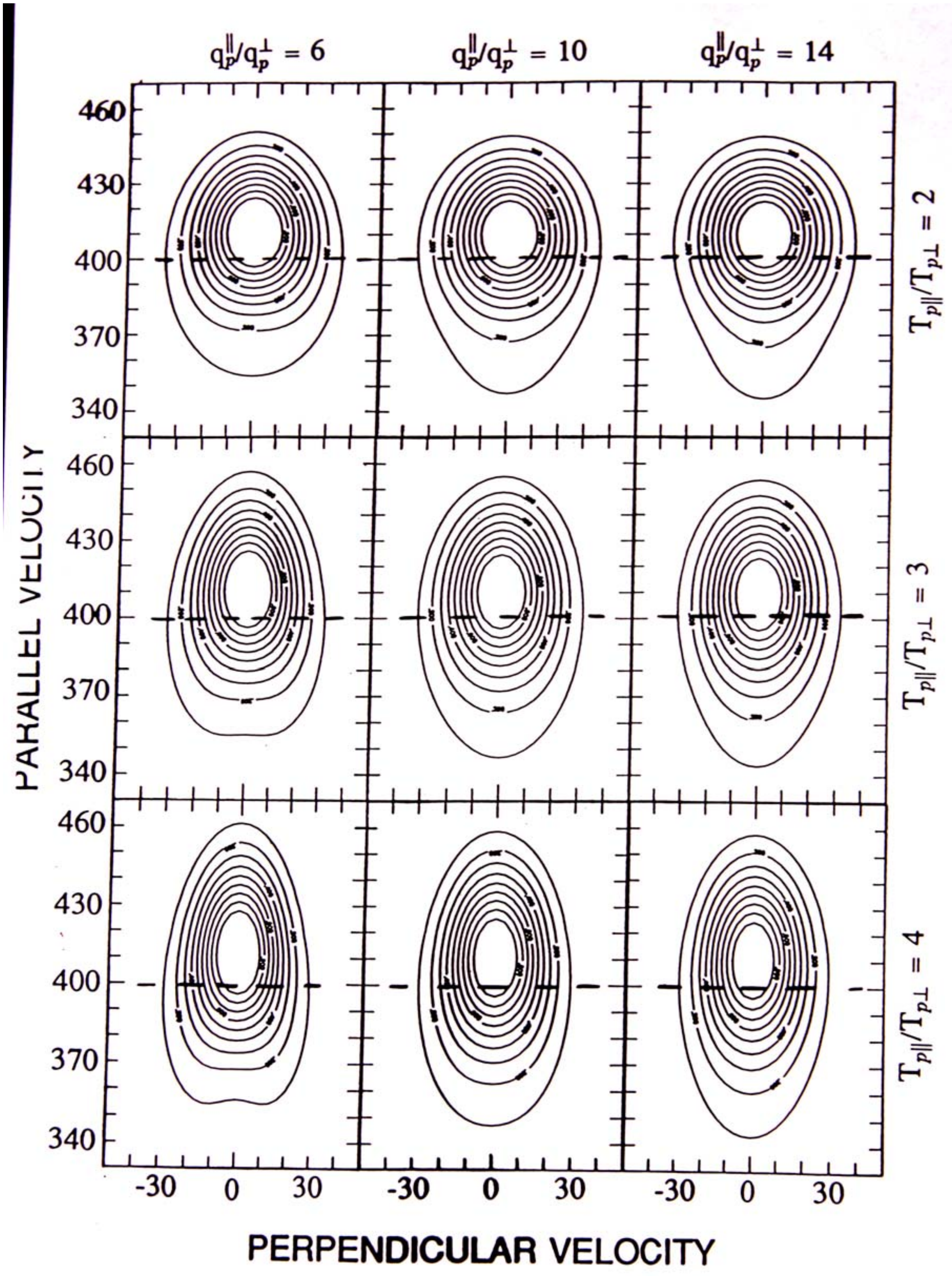
$$q_p = 0.15 \times 10^{-6} \quad \mathrm{erg\ cm^{-2}\ s^{-1}}$$

$$\begin{cases} q_p^{\parallel} = 0.1 \times 10^{-6} \quad erg\ \mathrm{cm}^{-2} s^{-1} \\ q_p^{\perp} = 0.1 \times 10^{-6} \quad erg\ \mathrm{cm}^{-2} s^{-1} \end{cases}$$

Vela 3
Proton f
1 A.U.







$q_p^{\parallel}/q_p^{\perp} = 6$
$q_p^{\parallel}/q_p^{\perp} = 10$
$q_p^{\parallel}/q_p^{\perp} = 14$
$T_{p\parallel}/T_{p\perp} = 2$
$T_{p\parallel}/T_{p\perp} = 3$
$T_{p\parallel}/T_{p\perp} = 4$
PARALLEL VELOCITY
460
430
400
370
340
-30
0
30
PERPENDICULAR VELOCITY

Conclusion

- Flow conditions determine which transport equations to adopt.
- Transport equations can describe far-from-equilibrium flows.
- The Bi-Maxwellian 16-moment equations are “valid” for the polar wind.
- Global models are expected to be based on a fluid-like approach.